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ISO(3, 1) Gauge Theory of Gravity

Shao Changgui[1] **and Xu Bangqing**[2]

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We establish a Poincaré affine frame bundle, give the operation of gauge group $ISO(3, 1)$ on the fiber bundle, obtain the action of the Poincaré gauge theory of gravity, and advance two sets of gravitational gauge field equations of the theory. We discuss some special cases of the theory.

1. POINCARÉ AFFINE FRAME BUNDLE

Denote the spacetime manifold by M, the Minkowski space by M'. Let $ISO(3, 1)$ be the inhomogeneous special orthogonal group of a metric of three spacelike and one timelike dimensions. Since M possesses local $ISO(3, 1)$ invariance, then $\forall x \in M$, there exists a set of local Poincaré affine frames $\{\lambda, e_i\}_{(x)}$. The union

$$P(M, ISO(3,1)) \equiv \bigcup_{x\in M} \{\lambda, e_i\}_{(x)} \qquad \text{for all} \quad x \in M$$

is the Poincaré affine frame bundle (Kobayashi and Nomizu, 1963). Here $\lambda = (\lambda^i)$, $i = 0, 1, 2, 3$, is a point in M'_x, the tangent space of M at $x \in M$: $T_x(M) = M'_x$. The $\{e_i\}$ is a local Lorentz moving frame on M.

If A is an element of group $ISO(3, 1)$, we may write the element A as

$$A = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix}$$

where

$$\xi = (\xi^i) \in M', \qquad a = (a^i_j) \in SO(3, 1)$$

[1]Wuhan Institute of Technology, China.
[2]Institute of Atomic Energy, P.O. Box 275, Academia Sinica, Beijing, China.

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Then $\forall u \in P(M, ISO(3,1))$, we may denote u by (x, A). Choose the action of group $ISO(3,1)$ on the Poincaré affine frame $\{\lambda, e_i\}$ as the right action. If $B \in ISO(3,1)$, the right action can be defined as

$$u' = uB, \qquad u' = (x, AB)$$

Here

$$u, u' = P(M, ISO(3,1))$$

$$B = \begin{pmatrix} b & \eta \\ 0 & 1 \end{pmatrix} \in ISO(3,1)$$

$$AB = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & \eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ab & a\eta + \xi \\ 0 & 1 \end{pmatrix} \in ISO(3,1)$$

2. ACTION OF *ISO*(3, 1) GAUGE THEORY OF GRAVITY

Let us denote the generators of the group $ISO(3,1)$ by $\{L_{ij}, P_k\}$, where L_{ij} are the generators of the group $SO(3,1)$ and P_k are the generators of the space-time translation group. For any $u = (x, a) \in P(M, ISO(3,1))$, we denote the tangent space of $P(M, ISO(3,1))$ at u by $T_u(P(M, ISO(3,1)))$; then on the bundle space $P(M, ISO(3,1))$

$$\{D_\mu, L_{ij}, P_k\} \tag{1}$$

is a noncoordinate basis in $T_u(P(M, ISO(3,1)))$. Here

$$D_\mu = \partial_\mu - \tfrac{1}{2}B(x)^{ij}_\mu L_{ij} - V(x)^k_\mu P_k \qquad (\mu, i = 0, 1, 2, 3)$$

where $B(x)^{ij}_\mu$ are the coefficients of the Lorentz connection and $V(x)^k_\mu$ are the Lorentz vierbein fields.

From the commutation relation $[D_\mu, D_\nu]$ we can obtain the expression

$$[D_\mu, D_\nu] = -\tfrac{1}{2}F^{ij}_{\mu\nu}L_{ij} - Q^k_{\mu\nu}P_k$$

where

$$F^{ij}_{\mu\nu} = \partial_\mu B^{ij} + B^i_{uk}B^{kj}_\mu - (\mu \leftrightarrow \nu)$$

is the curvature tensor of M and is defined in the moving frame, and

$$Q^k_{\mu\nu} = \partial_\mu V^i_\nu + B^i_{\mu j}V^j_{\nu\nu} - (\mu \leftrightarrow \nu)$$

is the torsion tensor of M and is defined in the same frame as above.

In the Poincaré gauge theory of gravity (PGTG), both $B(x)^{ij}_\mu$ and $V(x)^k_\mu$ are gauge potentials, and consequently both $F^{ij}_{\mu\nu}$ and Q^k_μ are gauge field strengths.

We now define a metric on bundle space $P(M, ISO(3,1))$ as follows:

$$\begin{aligned} G_{\mu\nu} &= \langle e_\mu, e_\nu \rangle = g_{\mu\nu} \\ G_{ij,kl} &= \langle L_{ij}, L_{kl} \rangle = g_{ij,kl} \\ G_{\mu,ij} &= \langle e_\mu, L_{ij} \rangle = 0 \\ G_{\mu k} &= \langle e_\mu, P_k \rangle = 0 \end{aligned} \tag{2}$$

Here $g_{\mu\nu} = V^i_\mu V^i_\nu \eta_{ij}$ is the metric on M and in the natural frame $\{e_\mu\}$, $\eta_{ij} = \mathrm{diag}(1, -1, -1, -1)$, and $g_{ij,kl} = \frac{1}{4} C^{pq}_{ij,mn} C^{mn}_{pq,kl}$ is the Cartan metric on the group $SO(3,1)$, where $C^{pq}_{ij,mn}$ are structure constants of the gauge group $SO(3,1)$.

If on the bundle space $P(M, ISO(3,1))$, $\{dx, B^{ij}, \theta^k\}$ is a dual basis of $\{D, L_{ij}, P_k\}$, then the square of the line element on $P(M, ISO(3,1))$ is

$$ds^2 = g_{\mu\nu}\, dx^\nu\, dx^\nu + \tfrac{1}{4} g_{ij,kl} B^{ij} \otimes B^{kl} + \eta_{mn} \theta^m \otimes \theta^n,$$

where $B^{ij} = B^{ij}_\mu\, dx^\mu$ is the Lorentz connection form on the bundle $P(M, ISO(3,1))$, and $\theta^k = V^k_\mu\, dx^\mu$ is the translation connection form on the bundle $P(M, ISO(3,1))$.

On the bundle space $P(M, ISO(3.1))$, defining the connection and calculating the curvature basis (1) and the metric (2), one finds the following curvature scalar:

$$\bar{R} = R + R_{ISO(3,1)} - \tfrac{1}{4} F_{\mu\nu j}^{\ \ \ \ i} F^{\mu\nu j}_{\ \ \ \ i} - \tfrac{1}{4} Q^k_{\mu\nu} Q^{\mu\nu}_k \tag{3}$$

where R is the Einstein curvature scalar of M; $R_{ISO(3,1)}$ is the curvature scalar of group $ISO(3,1)$ (arbitrary constant); $-\frac{1}{4} F_{\mu\nu j}^{\ \ \ \ i} F^{\mu\nu j}_{\ \ \ \ i}$ is the curvature kinetic energy term of the gauge fields; and $-\frac{1}{4} Q^k_{\mu\nu} Q^{\mu\nu}_k$ is the torsion kinetic energy term of the gauge fields. Thus the action of the $ISO(3,1)$ gauge theory of gravity can be written as

$$\begin{aligned} S &= \int \left(c\mathscr{L}_m V + RV + R_{ISO(3,1)} V - \frac{\rho}{4} V F_{\mu\nu j}^{\ \ \ \ i} F^{\mu\nu j}_{\ \ \ \ i} - \frac{\rho'}{4} V Q^k_{\mu\nu} Q^{\mu\nu}_k \right) d^4x \\ &= \int \left(c\mathscr{L}_m V + RV + R_{ISO(3,1)} V + \rho \mathscr{L}_g V + \rho' \mathscr{L}'_g V \right) d^4x \end{aligned} \tag{4}$$

where $\mathscr{L}_m = \mathscr{L}_m(\psi, \psi_{,\mu})$ is the Lagrangian of the matter field ψ, $V = \det(V^i_\mu) = (-g)^{1/2}$, $C = 8\pi k$ (k is the Newtonian gravitational constant), ρ and ρ' are gauge gravitational constants, and

$$\mathscr{L}_g = -\tfrac{1}{4} F_{\mu\nu j}^{\ \ \ \ i} F^{\mu\nu j}_{\ \ \ \ i}, \qquad \mathscr{L}'_g = -\tfrac{1}{4} Q^i_{\mu\nu} Q^{\mu\nu}_i$$

From expression (4) we know that the Lagrangian of the theory is

$$\mathscr{L} = C\mathscr{L}_m + R + R_{ISO(3,1)} + \mathscr{L}_g + \mathscr{L}'_g$$

The arbitrary constant $R_{ISO(3,1)}$ is nothing to gauge transformation on M; it will act as the cosmological term (Chang and Massouri, 1978) in the Lagrangian; we may choose $R_{ISO(3,1)}$ as zero. Then the expression (4) can be written as

$$S=\int\left(C\mathscr{L}_m V+RV-\frac{\rho}{4}VF_{\mu\nu j}^{\ \ i}F^{\mu\nu j}_{\ \ \ i}-\frac{\rho'}{4}VQ^k_{\mu\nu}Q^{\mu\nu}_k\right)d^4x$$

$$=\int (C\mathscr{L}_m V+RV+\rho\mathscr{L}_g V+\rho'\mathscr{L}'_g V)\, d^4x \tag{5}$$

3. FIELD EQUATIONS OF PGTG

In the Poincaré GTG, the matter field ψ, the vierbein fields $\bar{V}^i_{\ \mu}$, and the connection coefficients B^{ij}_μ all are kinetic variables of variational principle. For the Lagrangian $\mathscr{L}_m$ if we apply the variational principle under the variation of the matter field ψ, we obtain the Euler equation for the matter field:

$$\frac{\partial\mathscr{L}_m}{\partial\psi}-\frac{1}{V}\partial_\mu\left(V\frac{\partial\mathscr{L}_m}{\partial\psi_{,\mu}}\right)=0$$

For the action (5), applying the variational principle under the variation of the fields V^μ_i, we have

$$dS=\delta\int\left(C\mathscr{L}_m V+RV-\frac{\rho}{4}VE_{\mu\nu j}^{\ \ i}F^{\mu\nu j}_{\ \ \ i}-\frac{\rho'}{4}VQ^k_{\mu\nu}Q^{\mu\nu}_k\right)d^4x=0$$

Since

$$\frac{1}{V}\frac{\delta(\mathscr{L}_m V)}{\delta V^\mu_i}=T_{\lambda\nu}\frac{\delta g^{\lambda\nu}}{\delta V^\mu_i}=2T^i_{\ \mu}$$

$$\frac{1}{V}\frac{\delta(RV)}{\delta V^\mu_i}=2G^i_{\ \mu}$$

where $T_{\lambda\nu}$ is the mass tensor in the natural frame, $T^i_{\ \mu}$ is the mass tensor in the moving frame, and $G^i_{\ u}=R^i_{\ \mu}-\frac{1}{2}V^i_{\ \mu}R$ is the Einstein tensor in the moving frame, expression (6) can be written as

$$\delta S=\int (2CT^i_{\ \mu}+2G^i_{\ \mu})V\delta V^\mu_i\, d^4x$$

$$-\left(\frac{\rho}{4}\delta\int VF_{\mu\nu j}^{\ \ i}F^{\mu\nu j}_{\ \ \ i}+\frac{\rho'}{4}\delta\int VQ^k_{\mu\nu}Q^{\mu\nu}_k\right)d^4x=0 \tag{7}$$

Calculations yield

$$-\frac{\rho}{4}\delta\int VF_{\mu\nu j}{}^{i}F^{\mu\nu j}{}_{i}\,d^4x=-\rho\int V[\tfrac{1}{2}\operatorname{tr}(F_{\lambda\nu}F^{\lambda\nu})V^i_\mu-2\operatorname{tr}(F_{\mu\sigma}F^{\lambda\sigma})V^i_\mu]\delta V^\mu_i\,d^4x \tag{8}$$

$$-\frac{\rho'}{4}\delta\int VQ^i_{\mu\nu}Q^{\mu\nu}_i\,d^4x=\rho'\int V(\tfrac{1}{4}V^i_\lambda Q^k_{\mu\nu}Q^{\mu\nu}_k-Q^{\sigma\nu}_j Q^j_{\mu\nu}V^i_\sigma)\,\delta V^\mu_i\,d^4x \tag{9}$$

Using expressions (8) and (9) in (7), we get

$$R^i_\mu-\tfrac{1}{4}V^i_\mu R=-CT^i_\mu-\rho t^i_\mu-\rho'\tau^i_\mu \tag{10}$$

where

$$t^i_\mu=-\operatorname{tr}(F_{\mu\sigma}F^{\lambda\sigma})V^i_\lambda+\tfrac{1}{4}\operatorname{tr}(F_{\lambda\nu}F^{\lambda\nu})V^i_\mu \tag{11}$$

$$2\tau^i_\mu=-Q^{\lambda\nu}_j Q^j_{\mu\nu}V^i_\lambda+\tfrac{1}{4}Q^{\lambda\nu}_j Q^j_{\lambda\nu}V^i_\mu \tag{12}$$

In this gauge theory we define t^i_μ as the energy-momentum tensor of the gauge potential B^{ij}_μ, and τ^i_μ as that of the gauge potential V^i_μ.

Expression (10) represents a set of field equations of Poincaré GTG.

By varying the Lorentz connection coefficients B^{ij}_μ, we can obtain another set of field equations of the theory. The Euler equations are

$$C\frac{\partial(\mathcal{L}_m V)}{\partial B^{ij}_\mu}-C\partial_\nu\frac{\partial(\mathcal{L}_m V)}{\partial B^{ij}_{\mu,\nu}}+\frac{\partial(RV)}{\partial B^{ij}_\mu}-\partial_\nu\frac{\partial(RV)}{\partial B^{ij}_{\mu,\nu}}$$

$$+\rho\frac{\partial(\mathcal{L}_g V)}{\partial B^{ij}_\mu}-\rho\partial_\nu\frac{\partial(\mathcal{L}_g V)}{\partial B^{ij}_{\mu,\nu}}+\rho'\frac{\partial(\mathcal{L}'_g V)}{\partial B^{ij}_\mu}-\rho'\partial\nu\frac{\partial(\mathcal{L}'_g V)}{\partial B^{ij}_{\mu\nu}}=0 \tag{13}$$

After extensive calculations we find

$$\frac{1}{V}\left\{\frac{\partial(RV)}{\partial B^{ij}_\mu}-\partial_\nu\frac{\partial(RV)}{\partial B^{ij}_{\mu,\nu}}\right\}$$
$$=Q^\mu_{ij}-Q^\lambda_{ki}V^k_\lambda V^\mu_j-Q^\lambda_{jk}V^k_\lambda V^\mu_i$$
$$\equiv K^\mu_{ij}\quad\text{(cotorsion)} \tag{14}$$

$$\frac{1}{V}\left\{\frac{\partial(\mathcal{L}_g V)}{\partial B^{ij}_\mu}-\partial_\nu\frac{\partial(\mathcal{L}_g V)}{\partial B^{ij}_{\mu,\nu}}\right\}$$
$$=F^{\mu\nu}_{ij,\nu}-B^k_{\nu i}F^{\mu\nu}_{kj}-B^k_{\nu j}F^{\mu\nu}_{ik}-\{^{\lambda}_{\lambda\ \sigma}\}F^{\mu\sigma}_{ij}-\{^{\mu}_{\sigma\ \nu}\}F^{\sigma\nu}_{ij}\equiv F^{\mu\nu}_{ij\|\nu} \tag{15}$$

where $\|$ denotes the twofold covariant derivative in the natural and moving frames; and

$$\frac{1}{V}\left\{\frac{\partial(\mathcal{L}'_g V)}{\partial B^{ij}_\mu}-\partial_\nu\frac{\partial(\mathcal{L}'_g)}{\partial B^{ij}_{\mu,\nu}}\right\}=-Q^\mu_{ij} \tag{16}$$

where

$$Q^{\mu}_{i,j} = (V^k_{\nu,\lambda} - V^k_{\lambda,\nu} + B^k_{\mu\nu} - B^k_{\nu\mu}) V^\nu_i V^\lambda_j V^\mu_k, \qquad \partial_\nu \frac{\partial(RV)}{\partial B^{ij}_{\mu,\nu}} = 0 \tag{17}$$

Let

$$\frac{1}{V} \frac{\partial(\mathscr{L}_m V)}{\partial B^{ij}_\mu} \equiv S^\mu_{ij} \tag{18}$$

where S^μ_{ij} is the spin current of the matter field ψ, and use expressions (14)-(18) in (13); we then obtain the second set of field equations of Poincaré GTG:

$$CS^\mu_{ij} + K^\mu_{ij} - \rho' Q^\mu_{ij} = -\rho F^{\mu\nu}_{ij\|\nu} \tag{19}$$

Equation (19) relates the spin current, torsion, cotorsion, and change of field strength; it represents a set of new equations introduced into gravity with Poincaré GTG.

4. CONCLUSIONS

The field equations of Poincaré gravitational gauge theory consist of the following set:

$$R^i_\mu - \tfrac{1}{2} V^i_\mu R = -CT^i_\mu - \rho t^i_\mu - \rho' \tau^i_\mu \tag{10}$$

$$CS^\mu_{ij} + K^\mu_{ij} - \rho' Q^\mu_{ij} = -\rho F^{\mu\nu}_{ij\|\nu} \tag{19}$$

1. If the space-time manifold M is a Riemann space (torsion-free), then the gauge field equations become

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -CT_{\mu\nu} - \rho t_{\mu\nu} \tag{20}$$

$$CS^u_{ij} = -\rho F^\mu_{ij\|\nu} \tag{21}$$

Expressions (20) and (21) are field equations of PGTG in Riemannian space-time.

2. If M is a Riemann space and we ignore the interaction with gravitational gauge fields, the $\mathscr{L}_g$ and $\mathscr{L}'_g$ vanish in the Lagrangian $\mathscr{L}$; then equation (21) also vanishes and equation (20) degenerates to the Einstein equation:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -CT_{\mu\nu} \tag{22}$$

3. If M is a Riemann-Cartan space and we ignore the terms $\mathscr{L}_g$ and $\mathscr{L}'_g$ in the Lagrangian $\mathscr{L}$, equations (10) and (19) become

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -CT_{\mu\nu} \tag{23}$$

$$CS^\mu_{ij} + K^\mu_{ij} = 0 \tag{24}$$

Evidently the Einstein equations are special versions of the field equations of Poincaré GTG for no gauge and no torsion. Equations (23) and (24) are the field equations obtained from Poincaré GTG for torsion but no gauge. The theory using equations (23) and (24) can provide a generalized Einstein-Cartan theory for gravity.

4. If the group $ISO(3, 1)$ degenerates into its subgroup $SO(3, 1)$, the generators P_k vanish and the principal fiber bundle established in Section 1 becomes the Lorentz frame bundle:

$$P(M, SO(3,1)) \equiv \bigcup_{x\in M} \{e_i\}_{(x)} \qquad \text{for all} \quad x \in M.$$

In this case the theory obtained is the Lorentz gauge theory of gravity (Massouri and Chang, 1976); the Lagrangian is

$$\mathscr{L} = C\mathscr{L}_m + R + R_{so(3,1)} + \mathscr{L}_g$$

and the field equations are

$$R^i_{\ \mu} - \tfrac{1}{2}V^i_{\ \mu}R = -CT^i_{\ \mu} - t^i_{\ \mu} \tag{25}$$

$$CS^{\mu}_{ij} + K^{\mu}_{ij} = -F^{\mu\nu}_{ij\|\nu} \tag{26}$$

5. In the natural frame the set of equations (10) and (19) becomes

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -CT_{\mu\nu} - \rho t_{\mu\nu} - \rho'\tau_{\mu\nu} \tag{27}$$

$$CS^{\mu}_{\nu,\lambda} + K^{\mu}_{\nu\lambda} - \rho' Q^{\mu}_{\nu\lambda} = -\rho F^{\mu\rho}_{\nu\lambda\|\rho} \tag{28}$$

6. Equation (27) shows that the gravitational metric "potential" $g_{\mu\nu}$ can still be regarded as the fundamental physical quantity in the gauge theory, even though the gravitational metric potentials obtained with PGTG and GR are different from each other.

Because the energy-momentum tensor $t^i_{\ \mu}$ is constructed from the square of the curvature and $\tau^i_{\ \mu}$ from the square of the torsion, the influence of the terms $-\rho t^i_{\ \mu}$ and $-\rho'\tau^i_{\ \mu}$ in equation (10) is small compared with the effect of the mass tensor $t_{\mu\nu}$; still, the emergence of the terms $\rho t^i_{\ \mu}$ and $-\rho'\tau^i_{\ \mu}$ in equation (10) is a new result introduced into gravity from PGTG.

Equation (19) represents a set of new gravitational field equations, establishing the relationship of matter spin, space-time torsion, and change of space-time curvature. These relations may have significance for the development of gravity theory.

REFERENCES

Chang, L. N., and Massouri, F. (1978). *Physical Review D* **17**, 3168.
Kobayashi, S. and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. I.
Massouri, F., and Chang, L. N. (1976). *Physical Review D* **13**, 3192.